Coherence and Phase-space II

VSSUP Lectures 2014

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Outline

Ultracold atoms - the ideal quantum system

ULTRALOW temperatures down to $1nK$

What is different about ultracold atoms?

- Atoms are trapped in a hard vacuum
- Cooling to nanoKelvins or less
- Can have either bosons or fermions
- Atom 'lasers' - atoms behave as quantum objects
- Correlations - mean field theory doesn't always work
- Dynamics - time-evolution is very important

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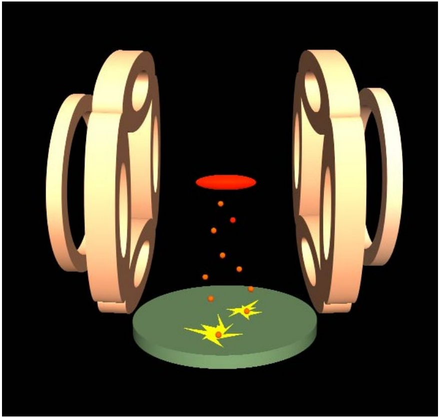
Typical experiment (Orsay, ANU)



How to calculate dynamics?

Classical solution: - use Hamilton's equations

$$\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \dot{q}_i &= \frac{\partial H}{\partial p_i} \end{aligned}$$

Quantum mechanics replaces classical quantities by corresponding operators with commutators, so that

$$\begin{aligned} [\widehat{q}_i, \widehat{p}_j] &= i\hbar\delta_{ij} \\ [\widehat{q}_i, \widehat{q}_j] &= [\widehat{p}_i, \widehat{p}_j] = 0 \end{aligned}$$

Then, for any operator $\hat{O}$, in the Heisenberg picture:

$$\frac{\partial \hat{O}}{\partial t} = \frac{1}{i\hbar}\left[\widehat{O}, \widehat{H}\right]$$

What about mixtures of states?

Suppose the quantum system is in a mixture of quantum states $|\psi_m\rangle$ with probability p_m . Then the density matrix $\hat{\rho}$ is defined as:

$$\hat{\rho} = \sum_m p_m |\psi_m\rangle \langle\psi_m|$$

In the Schroedinger picture, we let states evolve in time, not operators!

$$\frac{\partial\hat{\rho}}{\partial t} = \frac{1}{i\hbar}\left[\widehat{H},\widehat{\rho}\right]$$

Then, for any operator $\hat{O}$, the expectation value of the observable is:

$$\left\langle\hat{O}\right\rangle = Tr\left[\hat{\rho}\,\widehat{O}\right]$$

What is the Hamiltonian anyway?

What about the quantum fields with hats?

Here, $\widehat{\Psi}_i$ is a quantum field of spin-index i:

$$\left[\widehat{\Psi}_i(\mathbf{x}),\widehat{\Psi}_i^\dagger(\mathbf{x}')\right]_\pm = \delta_{ij}\delta^D(\mathbf{x}-\mathbf{x}')$$

In second quantization the quantum Hamiltonian is

$$\begin{aligned}\widehat{H} &= \sum_i\int d^D\mathbf{x}\left\{\frac{\hbar^2}{2m}\nabla\widehat{\Psi}_i^\dagger(\mathbf{x})\cdot\nabla\widehat{\Psi}_i(\mathbf{x})+V_i(\mathbf{x})\widehat{\Psi}_i^\dagger(\mathbf{x})\widehat{\Psi}_i(\mathbf{x})\right\}\\ &+ \sum_{ij}\frac{U_{ij}}{2}\int d^D\mathbf{x}\widehat{\Psi}_i^\dagger(\mathbf{x})\widehat{\Psi}_j^\dagger(\mathbf{x})\widehat{\Psi}_j(\mathbf{x})\widehat{\Psi}_i(\mathbf{x})\,.\end{aligned}$$

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$$\begin{aligned}\widehat{H} &= \sum_i \int d^D\mathbf{x}\left\{\frac{\hbar^2}{2m}\nabla\widehat{\Psi}_i^\dagger(\mathbf{x})\cdot\nabla\widehat{\Psi}_i(\mathbf{x}) + V_i(\mathbf{x})\widehat{\Psi}_i^\dagger(\mathbf{x})\widehat{\Psi}_i(\mathbf{x})\right\} \\ &+ \sum_{ij}\frac{U_{ij}}{2}\int d^D\mathbf{x}\widehat{\Psi}_i^\dagger(\mathbf{x})\widehat{\Psi}_j^\dagger(\mathbf{x})\widehat{\Psi}_j(\mathbf{x})\widehat{\Psi}_i(\mathbf{x})\,.\end{aligned}$$

What are the parameters?

This describes a dilute gas at low enough temperatures,

- $\langle\widehat{\Psi}_i^\dagger(\mathbf{x})\widehat{\Psi}_i(\mathbf{x})\rangle$ is the spin i atomic density,
- m is the atomic mass,
- V_i is the atomic trapping potential & Zeeman shift
- U_{ij} is related to the S-wave scattering length in three dimensions by:

$$U_{ij} = \frac{4\pi\hbar^2 a_{ij}}{m}.$$

- Here we implicitly assume a momentum cutoff $k_c << 1/a$

Simplest method for state evolution

Suppose the quantum system is described by a few modes:

$$|\psi\rangle = \sum \psi_{\mathbf{N}} |N_1, N_2, \ldots N_m\rangle = \sum \psi_{\mathbf{N}} |\mathbf{N}\rangle$$

Then, let $H_{\mathbf{NM}} = \langle \mathbf{N}| \hat{H} |\mathbf{M}\rangle$ and: $\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}\hat{H}|\psi\rangle$
Hence, we have a simple matrix equation:

$$\frac{d}{dt}\psi_{\mathbf{N}} = -\frac{i}{\hbar}\sum_{\mathbf{M}} H_{\mathbf{NM}}\psi_{\mathbf{M}}$$

(4) Prove the last equation using orthogonality

Problem: quantum theory is exponentially complex!

Quantum many-body problems are very large

- consider N particles distributed among M modes
- take $N \simeq M \simeq 500,000$:
- Number of quantum states: $N_s = 2^{2N} = 2^{1,000,000}$
- **More quantum states than atoms in the universe**
- How big is your computer?
- **Can't diagonalize $2^{1,000,000} \times 2^{1,000,000}$ Hamiltonian!**

What about losses and damping?

Damping can be treated using a master equation

- The density matrix $\hat{\rho}$ evolves as:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[\hat{H},\hat{\rho}\right] + \sum_j \kappa_j \int d^3\mathbf{r}\mathscr{L}_j\left[\hat{\rho}\right]$$

- Here the Liouville terms describe coupling to the reservoirs:

$$\mathscr{L}_j\left[\hat{\rho}\right] = 2\hat{O}_j\hat{\rho}\,\hat{O}_j^\dagger - \hat{O}_j^\dagger\hat{O}_j\hat{\rho} - \hat{\rho}\,\hat{O}_j^\dagger\hat{O}_j$$

- For n-particle collisions: $\hat{O}_i = \left[\widehat{\Psi}_i(\mathbf{r})\right]^n$

Traditional quantum theory methods?

- numerical diagonalisation?
  intractable for $\gtrsim 10$ modes
- operator factorization
  not applicable for strong correlations
- perturbation theory
  diverges at strong couplings
- exact solutions
  not applicable for quantum dynamics

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Quantum theory in classical phase-space

Properties of Wigner/Moyal phase-space

- Maps quantum states into **classical phase-space** $\alpha = p + ix$
- **Wigner first published this representation**
- Moyal showed equivalence to quantum mechanics
- Complexity grows only linearly with number of modes!

Problem: Wigner distribution can have negative values

- **Need to truncate equations to get positive probabilities**

Detailed equivalence

Mapping of characteristic functions

$$W(\boldsymbol{\alpha}) = \frac{1}{\pi^{2M}} \int d^{2M}\mathbf{z} \left\langle e^{i\mathbf{z}\cdot(\hat{\mathbf{a}}-\boldsymbol{\alpha})+i\mathbf{z}^*\cdot\left(\hat{\mathbf{a}}^\dagger-\boldsymbol{\alpha}^*\right)} \right\rangle$$

Operator mean values

- $\left\langle \hat{a}_i^{\dagger m} \hat{a}_j^n \right\rangle_{SYM} = \int d^{2M}\boldsymbol{\alpha}\, \alpha_i^{*m} \alpha_j^n W(\boldsymbol{\alpha}) = \left\langle \alpha_i^{*m} \alpha_j^n \right\rangle_W$
- $\langle \hat{a}_j \rangle = \langle \alpha_j \rangle_W$
- $\left\langle \hat{a}_i^\dagger \hat{a}_j + \hat{a}_i \hat{a}_j^\dagger \right\rangle / 2 = \langle \alpha_i^* \alpha_j \rangle_W$

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Dynamical equivalence

Mapping of dynamical equations

$$\frac{\partial W(\boldsymbol{\alpha})}{\partial t} = \frac{1}{\pi^{2M}} \int d^{2M}\mathbf{z}\, Tr\left[\frac{\partial \hat{\rho}}{\partial t} e^{i\mathbf{z}\cdot(\hat{\mathbf{a}}-\boldsymbol{\alpha})+i\mathbf{z}^{*}\cdot(\hat{\mathbf{a}}^{\dagger}-\boldsymbol{\alpha}^{*})}\right]$$

Operator mappings

- $\hat{a}_j\hat{\rho} \rightarrow \left(\alpha_j + \frac{1}{2}\frac{\partial}{\partial\alpha_j^*}\right) W$
- $\hat{\rho}\,\hat{a}_j^\dagger \rightarrow \left(\alpha_j^* + \frac{1}{2}\frac{\partial}{\partial\alpha_j}\right) W$
- $\hat{a}_j^\dagger\hat{\rho} \rightarrow \left(\alpha_j^* - \frac{1}{2}\frac{\partial}{\partial\alpha_j}\right) W$
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Example: Wigner function for a coherent state

Suppose we have a single-mode BEC in a coherent state

$$\hat{\rho} = |\alpha_0\rangle \langle\alpha_0|$$

Hence:

$$W(\alpha) = \frac{1}{\pi^2} \int d^2z \langle\alpha_0| e^{iz\cdot(\hat{a}-\alpha)+iz\cdot\left(\hat{a}^\dagger-\alpha^*\right)} |\alpha_0\rangle$$

Solution with a little algebra

- $$W(\alpha) = \frac{2}{\pi} e^{-2|\alpha-\alpha_0|^2}$$

(5): show that this solution gives $\langle\alpha^*\alpha\rangle = 1/2$ for a vacuum state

Example: time-evolution of harmonic oscillator

Consider the harmonic oscillator

$$\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}$$

$$\frac{\partial \hat{\rho}}{\partial t} = -i\omega \left[\hat{a}^\dagger \hat{a}\widehat{\rho} - \widehat{\rho}\hat{a}^\dagger \hat{a}\right]$$

Operator mappings

- $\hat{a}^\dagger \hat{a}\hat{\rho} \to \left(\alpha^* - \frac{1}{2}\frac{\partial}{\partial \alpha}\right)\left(\alpha + \frac{1}{2}\frac{\partial}{\partial \alpha^*}\right) W$
- $\hat{\rho}\hat{a}^\dagger \hat{a} \to \left(\alpha - \frac{1}{2}\frac{\partial}{\partial \alpha^*}\right)\left(\alpha^* + \frac{1}{2}\frac{\partial}{\partial \alpha}\right) W$
- $$\frac{\partial W}{\partial t} = i\omega\left(\frac{\partial}{\partial \alpha}\alpha - \frac{\partial}{\partial \alpha^*}\alpha^*\right) W$$

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Operator mappings

- $\hat{a}^{\dagger}\hat{a}\hat{\rho} \rightarrow \left(\alpha^{*} - \frac{1}{2}\frac{\partial}{\partial\alpha}\right)\left(\alpha + \frac{1}{2}\frac{\partial}{\partial\alpha^{*}}\right)W$
- $\hat{\rho}\hat{a}^{\dagger}\hat{a} \rightarrow \left(\alpha - \frac{1}{2}\frac{\partial}{\partial\alpha^{*}}\right)\left(\alpha^{*} + \frac{1}{2}\frac{\partial}{\partial\alpha}\right)W$
- $$\frac{\partial W}{\partial t} = i\omega\left(\frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\alpha^{*}}\alpha^{*}\right)W$$

Harmonic oscillator solution

General result for harmonic oscillator

$$\frac{\partial W}{\partial t} = i\omega \left(\frac{\partial}{\partial \alpha} \alpha - \frac{\partial}{\partial \alpha^*} \alpha^* \right) W$$

Solution by method of characteristics

- $$\frac{\partial \alpha}{\partial t} = -i\omega\alpha$$
- $$\alpha(t) = \alpha(0) e^{-i\omega t}$$

(6): Prove this!

Fokker-Planck equations

Result of operator mappings:

$$\frac{\partial W}{\partial t} = \left\{ -\frac{\partial}{\partial \alpha_i} A_i + \frac{1}{2} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j^*} D_{ij} + \frac{1}{6} \frac{\partial^3}{\partial \alpha_i \partial \alpha_j^* \partial \alpha_k^*} T_{ijk} + \ldots \right\} W$$

Scaling to eliminate higher-order terms

$$x = \alpha / \sqrt{N}$$

$$\frac{\partial W}{\partial t} = \left\{ -\frac{1}{\sqrt{N}} \frac{\partial}{\partial x_i} A_i + \frac{1}{2N} \frac{\partial^2}{\partial x_i \partial x_j} D_{ij} + O\left(\frac{1}{N^{3/2}} \right) \right\} W$$

Stochastic equation

Result of operator mappings + truncation - valid if $N/M >> 1$:

$$\frac{\partial W}{\partial t} = \left\{ -\frac{\partial}{\partial \alpha_i} A_i + \frac{1}{2} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j^*} D_{ij} \right\} W$$

Equivalent stochastic equation

$$\frac{\partial \alpha_i}{\partial t} = A_i + \zeta_i(t)$$

where:

$$\left\langle \zeta_i(t) \zeta_j^*(t) \right\rangle = D_{ij} \delta\left(t - t'\right)$$

Example: BEC case

Result of operator mappings + truncation - for the GPE:

$$\frac{d\psi_j}{dt} = iK_j\psi_j - iU_{ij}|\psi_i|^2\psi_j - \gamma_j\psi_j + \sqrt{\gamma_j}\zeta_j(\mathbf{x},t)$$

Here the linear unitary evolution of the wave-function, is described by:

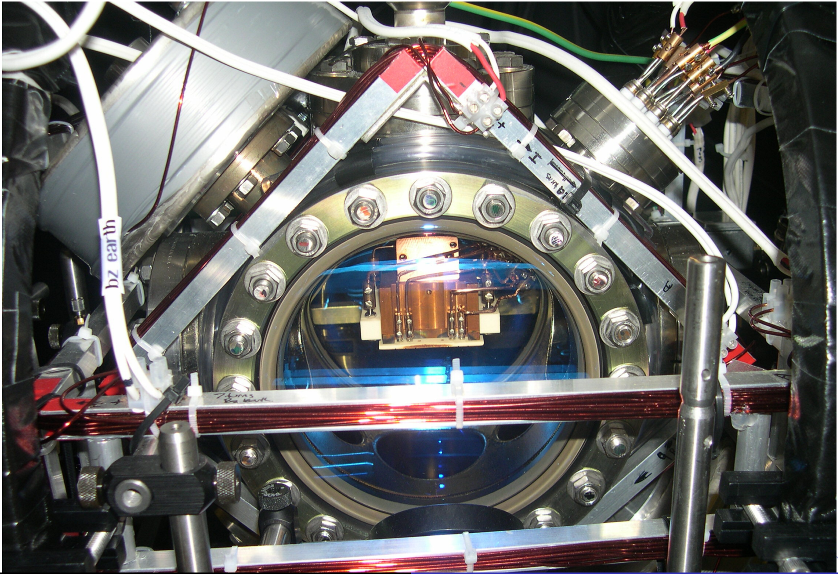
$$K_j = \hbar\nabla^2/2m - V_j(\mathbf{r})$$

while $\zeta_i(\mathbf{x},t)$ is a complex, stochastic delta-correlated Gaussian noise with

$$\left\langle \zeta_i(\mathbf{x},t)\zeta_j^*(\mathbf{x}',t')\right\rangle = \delta_{ij}\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(t-t'\right).$$

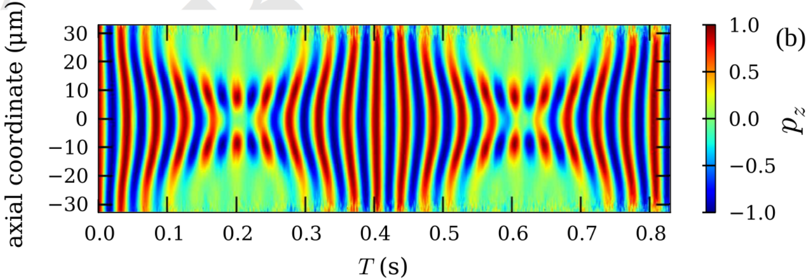
Initial fluctuations: $\langle\Delta\Psi_s(\mathbf{x})\Delta\Psi_u^*(\mathbf{x}')\rangle = \frac{1}{2}\delta_{su}\delta^3(\mathbf{x}-\mathbf{x}')$

Interferometry on an atom chip



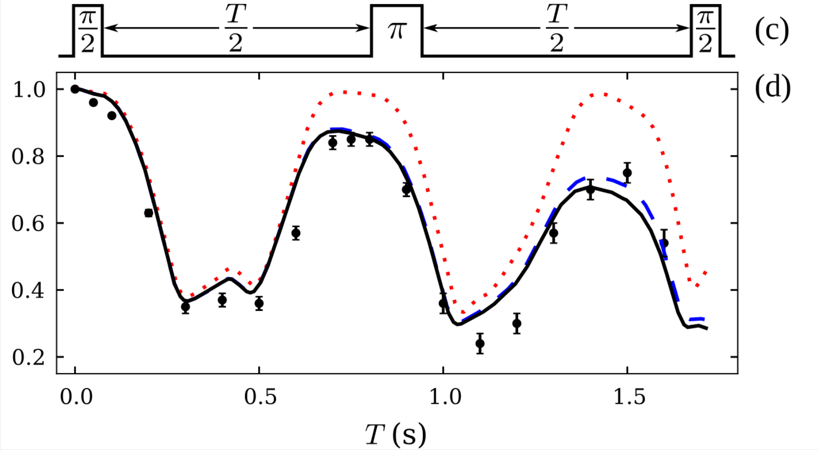
Interferometry

A two-component ^{87}Rb BEC is in a harmonic trap with internal Zeeman states $|1, -1\rangle$ and $|2, 1\rangle$, which can be coupled via an RF field.



Wigner simulations vs BEC fringe visibility

Blue line = Wigner simulation, black line = Wigner + local oscillator noise, red dots = GPE, error bars are measured



SUMMARY

Phase-space representation methods have many applications

Wigner phase-space is relatively simple!

- Maps **quantum field evolution** into a stochastic equation
- Can also be used to treat interferometry
- Advantage: No exponential complexity issues!
- Mathematical challenge:
  - truncation error needs to be checked: SEE Lecture 3!

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